

## GROWTH OF $N$ -DIMENSIONAL SPHERICAL BUBBLE WITHIN VISCOUS, SUPERHEATED LIQUID Analytical Solution

by

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*In this paper, we present the study of the behavior of spherical bubble in  $N$ -dimensions fluid. The fluid is a mixture of vapor and superheated liquid. The mathematical model is formulated in  $N$ -dimensions fluid on the basis of continuity and momentum equations, and solved its analytically. The variable viscosity is taken into an account problem. The obtained results show that the radius of bubble increases with the decreasing of the value of  $N$ -dimensions.*

**Key words:** *spherical bubbles,  $N$ -dimensions fluid, analytical solution, growth problem vapor and superheated liquid*

### Introduction

The dynamics of a growth or a collapse of vapor/gas bubbles are an area of great interest with several applications of science as dynamics of fluids, biomedical physics, and engineering and bioengineering applications [1-5]. The dynamics of vapor/gas bubbles are strongly dependent on many physical parameters such as the parameter of pressure of the vapor/gas that contained on it. In principle, these quantities must be determined. These determinations represent the solution of the conservation equations like continuity, momentum, and energy equations inside and outside the bubble joined together by suitable boundary conditions at the bubble interface.

Rayleigh [6] considers the first one who derived the Rayleigh equation for the dynamics of a gas-filled cavity, where this work [6] presented without consideration thermal effect, viscosity of liquid and surface tension. In addition to that, there are many efforts that they developed the bubble problem, as Plesset and Zwick [7] who studied the Rayleigh equation in a large wide by including viscosity of liquid and surface tension. The Rayleigh-Plesset equation describes the models dynamics of an empty or gas/vapor filled spherical cavity in an infinite volume of a liquid [7]. It is considered that the Rayleigh-Plesset equation is a special case of the Navier-Stokes equation, that describes the bubble dynamics in an infinite incompressible fluid. In general, there is no known general solution to the Rayleigh-Plesset equation. The equations are often solved numerically, but there are some efforts and attempts to solve it analytically in a special approximation like efforts in [6-13], *i. e.*, in [13] the solution

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of Rayleigh-Plesset equation is investigated analytically by using the transformation of Cole-Hopf in the bubble growth. Also, in another work [14, 15], authors studied the analytical approximation of Rayleigh equation in the collapse of an empty spherical bubble. For our knowledge, there are no generalized and analytical solutions of Rayleigh-Plesset equation in  $N$ -dimensions for vapor bubble growth where  $N$ -dimensions bubble is the generalized solutions of Rayleigh-Plesset equation in  $N$ -dimensions fluid and estimate the bubble radius for any value of  $N$ -dimensions. It is considered that the Rayleigh-Plesset equation in  $N$ -dimensions is satisfied at  $N \geq 3$ .

The main aim of the present work deals to study the analytical solution of the mathematical model of bubble dynamics in  $N$ -dimensions. The variable viscosity is taken into account. We interest in knowing the mechanism of bubble behavior in  $N$ -dimensions and compare the presented results with the previous available studies.

In this work we concern to formulate the mathematical model of the growth of bubble in the  $N$ -dimensions fluid that depends on the basis of Navier-Stokes equation. When we study the Navier-Stokes equation in  $N$ -dimension [16], we should take under our considerations the regularity criteria established [17] that have been used to set-up the bubble dynamics problem. The present work is to derive the Rayleigh-Plesset equation in  $N$ -dimensions fluid. This equation describes the growth of vapor bubble in  $N$ -dimensions. The analytical solution of the mathematical model is solved analytically by using the modifications of Plesset and Zwick [2], and Mohammadein [8]. We also take into account a surface tension and viscosity in the current work.

### Mathematical formulation

In the current section we introduce the derivation of the main equations as continuity equation and momentum equation in  $N$ -dimensions spherical fluid, that describe the problem of bubble dynamics in  $N$ -dimensions.

#### Continuity equation in $N$ -dimensions fluid

The continuity equation of liquids has the form:

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = 0 \quad (1)$$

where  $\vec{u}$  is the vector of velocity and  $\rho$  – the density of liquid. In the case of incompressible liquid that means  $\rho = \text{constant}$ , and for the case of spherically symmetric liquid in  $N$ -dimensions, the equation of continuity (1) takes the form:

$$\text{div}(\vec{u}) = \frac{1}{r^{N-1}} \frac{d}{dr} (r^{N-1} u) = 0 \quad (2)$$

where  $N$  represents the  $N$ -dimensions of fluid, and  $r$  – the radial co-ordinate. Note that, the point of radial co-ordinate,  $r$ , equals zero ( $r = 0$ ), that corresponds to the center of bubble.

The equation of vapor continuity in terms of the velocities reads as:

$$\rho_v \dot{R} = \rho(u)_{r=R} \quad (3)$$

where  $\rho_v$  is the density of vapor in flow,  $\dot{R}$  – the velocity of bubble wall,  $(u)_{r=R}$  – the velocity of liquid at the bubble boundary which can be expressed in the following form:

$$(u)_{r=R} = \varepsilon \dot{R} \quad (4)$$

where  $\varepsilon = 1 - \rho_v/\rho$ .

Integrating (2) w.r.t.  $r$ , the solution takes:

$$ur^{N-1} = c_1(t) \quad (5)$$

where  $c_1(t)$  is a function of time and it is independent with  $r$ . At  $r = R$  and using eq. (4) then  $c_1(t) = \varepsilon \dot{R} R^{N-1}$ , the velocity in the mixture of the vapor and superheated liquid and in  $N$ -dimensions becomes

$$u = \frac{\varepsilon \dot{R} R^{N-1}}{r^{N-1}} \quad (6)$$

where  $u$  is the mixture velocity in  $N$ -dimensions.

#### *Derivation of Rayleigh-Plesset equation in $N$ -dimensions fluid*

The equation of Navier-Stokes in spherical co-ordinates for the case of spherically symmetric and incompressible fluid has the form:

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} \right) = - \frac{\partial P'_l}{\partial r} + \eta \left[ \nabla^2 u - \frac{(N-1)u}{r^2} \right] \quad (7)$$

where  $u$  is the velocity of mixture of vapor and superheated liquid in  $N$ -dimensions fluid,  $\eta$  – the viscosity, and  $t$  – the time. The  $P'_l$  can be defined:

$$P'_l = P_l + 2\eta \frac{\partial u}{\partial r} \quad (8)$$

To evaluate the term  $\nabla^2 u - \{(N-1)u/r^2\}$  in eq. (9), we use this relation  $(1/r^{N-1})d/dr[(r^{N-1}u)]$ , then we get:

$$\nabla^2 u - \frac{(N-1)u}{r^2} = 0 \quad (9)$$

Differentiating eq. (8) w.r.t.  $r$ , and using eq. (6), the term  $\partial P'_l/\partial r$  can be rewritten in the form:

$$\frac{\partial P'_l}{\partial r} = \frac{\partial P_l}{\partial r} - 2\eta N(N-1)\varepsilon \frac{\dot{R} R^{N-1}}{r^{N+1}} \quad (10a)$$

Also

$$\frac{\partial u}{\partial t} = \frac{\varepsilon}{r^{N-1}} [\ddot{R} R^{N-1} + (N-1)R^{N-2}\dot{R}^2] \quad (10b)$$

and

$$u \frac{\partial u}{\partial r} = -\varepsilon(N-1) \frac{[R^{2(N-2)}]\dot{R}^2}{r^{(2N-1)}} \quad (10c)$$

Substituting from eqs. (9) and (10) into eq. (7), we get:

$$\rho \left\{ \frac{\varepsilon}{r^{N-1}} [\ddot{R} R^{N-1} + (N-1)R^{N-2}\dot{R}^2] - \varepsilon(N-1) \frac{R^{2(N-2)}\dot{R}^2}{r^{2N-1}} \right\} = - \frac{\partial P_l}{\partial r} + 2\eta N(N-1)\varepsilon \frac{\dot{R} R^{N-1}}{r^{N+1}} \quad (11)$$

Assuming that the vapor bubble is in an infinite sea of liquid and integrate eq. (11) w.r.t.  $r$  from  $r = R$  to  $r = \infty$ , then we get:

$$\left\{ \frac{1}{-(N-2)} \left[ \ddot{R} R^{N-1} + (N-1) R^{N-2} \dot{R}^2 \right] \frac{1}{r^{N-2}} \right|_{r=R}^{\infty} - \frac{(N-1)}{2(N-1)} \frac{R^{2(N-2)} \dot{R}^2}{r^{2N-1}} \right|_{r=R}^{\infty} \left\} = -\frac{1}{\varepsilon \rho} P_l \Big|_{r=R}^{\infty} + \frac{2\eta N(N-1)}{N\rho} \dot{R} R^{N-1} \frac{1}{r^N} \Big|_{r=R}^{\infty} \quad (12)$$

Note that,  $P_l \Big|_{r=R}^{\infty} = P_{\infty} - (\Delta P_l)_{r=R}^{\infty}$ .

In the Pressure balance at the bubble surface, A simple force balance at the surface of spherical bubble (at  $r = R$ ) is defined as  $\pi R^2 [P_v - (\Delta P_l)_{r=R}^{\infty}] = 2\pi R \sigma$  and the pressure at the bubble boundary (an ambient pressure) takes the form:

$$(\Delta P_l)_{r=R}^{\infty} = P_v - \frac{2\sigma}{R} \quad (13)$$

After some calculations, eq. (12) can be put in the form:

$$\frac{1}{(N-2)} R \ddot{R} + \frac{N}{2(N-2)} \dot{R}^2 = \frac{1}{\varepsilon \rho} \left( P_v - P_{\infty} - \frac{2\sigma}{R} \right) - \frac{2(N-1)}{\rho} \frac{\dot{R}}{R} \eta \quad (14)$$

Equation (14) is called the modified of Rayleigh-Plesset equation in  $N$ -dimensions for the growth (or collapse) of bubble in a viscous liquid.

In eq. (14), we remark that  $N$  is the number of space dimensions, and  $N \geq 3$  satisfies in the Rayleigh-Plesset equation in  $N$ -dimensions. Also, we note that we do not use the case of  $N = 2$  in our study, since eq. (14) contains logarithmic terms and, therefore, are singular. Now, the main aim is to find the generalized solutions of eq. (14). It seems impossible to construct solutions of eq. (14) by their direct method. However, it is possible to solve it by using the method that will introduce at next section.

### Method of the solution

To complete the study of bubble dynamics, we suppose that the vapor bubble is a spherical shape. The liquid is a mixture of vapor and superheated liquid where the mixture is in an incompressible. The pressure inside the bubble is to be uniform. The distribution of vapor density inside the bubble also is to be uniform except for a thin boundary-layer near the bubble wall. Any gravitational effects are omitted in the equation of motion. We take the viscosity into an account. So, let us to start, suppose the viscosity of the liquid reads

$$\eta = \frac{C R(t)}{\dot{R}(t)} \quad (15)$$

where  $C = \eta_0 \dot{R}_0 / R_0$ , where  $R$  is the ambient radius of bubble,  $\dot{R}_0$  – the initial velocity of bubble, and  $\eta_0$  – the initial viscosity.

In our study, we note that the pressure inside the bubble is  $P_v$ , and the pressure  $P_{\infty}$  in the liquid is constant. Assume that the difference of pressure  $\Delta P$  is constant in time, and should be  $P_v > P_{\infty}$ .

In order to complete the solution eq. (14) we assume the difference between the pressure at the bubble boundary and the pressure in the liquid  $\Delta P = P_v - P_\infty$  takes:

$$\Delta P = A(T_R - T_\infty) \quad (16)$$

where  $T_R$  is the temperature inside the bubble,  $T_\infty$  – the temperature in the liquid, and  $A$  – the constant, will be estimated in below.

Combining eqs. (14)-(16), eq. (14) becomes:

$$\frac{1}{(N-2)} R \ddot{R} + \frac{N}{2(N-2)} \dot{R}^2 = \frac{1}{\varepsilon \rho} \left[ A(T_R - T_\infty) - \frac{2\sigma}{R} \right] - \frac{2(N-1)}{\rho} C \quad (17)$$

In the following the initial conditions, when the bubble starts to appear that the initial conditions becomes:

$$t = 0, \quad R(0) = R_0, \quad \dot{R}(0) = \dot{R}_0, \quad \ddot{R}(0) = 0, \quad T_R = T_0 \quad (18)$$

Applying the initial conditions in eq. (18) into eq. (17), then we can calculate  $A$ . After some simplification of calculations. eq. (17) reads:

$$R \ddot{R} + \frac{N}{2} \dot{R}^2 = \frac{2(N-2)\sigma}{\varepsilon \rho R_0} \left( \frac{\Delta T_R^*}{\Delta \theta_0} + 1 \right) \left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} - \frac{2(N-2)}{\varepsilon \rho} \frac{\sigma}{R} - \frac{2(N-1)(N-2)}{\rho} C \quad (19)$$

where  $\Delta T_R^* = T_R - T_0$  is defined in down, and  $\Delta \theta_0 = T_0 - T_\infty$  is the superheating liquid.

From previous equations, eq. (19) can rewrite in the form:

$$\frac{1}{d_1 R^{d_2-1} \dot{R}^{d_3-1}} \frac{d}{dt} (R^{d_2} \dot{R}^{d_3}) = \frac{2(N-2)\sigma}{\varepsilon \rho R_0} \left( \frac{\Delta T_R^*}{\Delta \theta_0} + 1 \right) \left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} - \frac{2(N-2)}{\varepsilon \rho} \frac{\sigma}{R} - \frac{2(N-1)(N-2)}{\rho} C \quad (20)$$

where  $N/2 = d_2/d_1$ ,  $d_1 = d_3$ .

The parameter  $\Delta T_R^*$  is defined by Plesset *et al.* [7] in the form:

$$\Delta T_R^* = - \left( \frac{a_l}{\pi} \right)^{1/2} \int_0^t \frac{R^2(y_1) \left( \frac{\partial T}{\partial r} \right)_{r=R(y_1)}}{\left[ \int_0^t R^4(y_2) dy_2 \right]^{1/2}} dy_1 \quad (21)$$

where  $a_l$  is the thermal diffusivity.

Equation (21) represents a solution of the heat equation. Plesset and Zwick [7] achieve under the assumption where the bubble motion is sufficiently rapid because of any translational motion may be omitted which leads to vanish the convection term in heat equation and using the thin layer approximation around the growing bubble with error less than 10% [18].

To solve the system, we convert the system to dimensionless:

$$\begin{aligned}\chi &= \left( \frac{R}{R_0} \right)^3 \\ \nu &= \frac{\gamma}{R_0^4} \int_0^t R^4(y_2) dy_2 \\ \gamma &= \left( \frac{2\sigma}{\rho R_0^3} \right)^{1/2}\end{aligned}\quad (22)$$

The gradient of temperature in the mixture of vapor and superheated liquid at the bubble boundary reads:

$$(4\pi R^2)k_l \left( \frac{\partial T}{\partial r} \right)_{r=R} = \frac{d}{dt} \left[ \rho_v \left( \frac{4}{3} \pi R^3 \right) L \right] \quad (23)$$

where  $L$  is the latent heat,  $k_l$  – the thermal conductivity, and  $\rho_v$  – the density of vapor.

By using eq. (22) into eq. (23), then:

$$\left( \frac{\partial T}{\partial r} \right)_{r=R} = \frac{\gamma \rho_v L R_0}{3k_l} \chi^{2/3} \chi' \quad (24)$$

Substituting from eq. (22) into the R.H.S of eq. (20), then:

$$\frac{1}{d_1 R^{d_2-1} \dot{R}^{d_3-1}} \frac{d}{dt} (R^{d_2} \dot{R}^{d_3}) = \frac{\gamma^2 R_0^2}{3d_1} \frac{1}{\chi^{1/3(d_2+2d_3-7)} \chi^{d_3-1}} \frac{d}{d\nu} (\chi^{1/3(d_2+d_3)} \chi'^{d_3}) \quad (25)$$

After some simplification calculation  $\Delta T_R^*$  will be taken:

$$\Delta T_R^* = -\frac{\rho_v L R_0}{k_l} \left( \frac{\gamma a_l}{9\pi} \right)^{1/2} \int_0^\nu \frac{\chi'(\xi)}{(\nu - \xi)^{1/2}} d\xi \quad (26)$$

From eqs. (22), (25), and (26) into eq. (20), then we get:

$$\begin{aligned}\frac{\gamma^2 R_0^2}{3d_1} \frac{1}{\chi^{1/3(d_2+2d_3-7)} \chi^{d_3-1}} \frac{d}{d\nu} (\chi^{1/3(d_2+d_3)} \chi'^{d_3}) &= \frac{2(N-2)\sigma}{\varepsilon \rho R_0} \left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} - \\ &- \frac{2(N-2)\sigma}{\varepsilon \rho R_0} \left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \cdot \\ &\cdot \frac{\rho_v L R_0}{k_l \Delta \theta_0} \left( \frac{\gamma a_l}{9\pi} \right)^{1/2} \int_0^\nu \frac{\chi'(\xi)}{(\nu - \xi)^{1/2}} d\xi - \frac{2(N-2)}{\varepsilon \rho} \frac{\sigma}{R_0 \chi^{1/3}} - \frac{2(N-1)(N-2)}{\rho} C\end{aligned}\quad (27)$$

Dividing eq. (27) by:

$$\gamma^2 R_0^2 \left[ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left( \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right) \right]$$

and make some calculations, then eq. (27):

$$\begin{aligned} & \frac{1}{3d_1 \left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\}} \frac{1}{\chi^{1/3(d_2+2d_3-7)} \chi^{d_3-1}} \frac{d}{d\nu} \left[ \chi^{1/3(d_2+d_3)} \chi^{d_3} \right] = \frac{(N-2)}{\varepsilon} - \\ & - \lambda \int_0^\nu \frac{\chi'(\xi)}{(\nu-\xi)^{1/2}} d\xi - \frac{(N-2)}{\varepsilon \left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\}} \chi^{1/3} - \\ & - \frac{(N-1)(N-2)R_0}{\left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \sigma} C \end{aligned} \quad (28)$$

where

$$\lambda = \frac{2(N-2)\sigma}{\varepsilon \rho R_0^2} \frac{\rho_v L}{k_l \Delta \theta_0} \left( \frac{a_l}{9\pi \gamma^3} \right)^{1/2}$$

The L.H.S of eq. (28) represents essentially the acceleration effects of the bubble growth in the liquid. As the bubble grows, this acceleration tends towards zero, see [8], at complete growth under these conditions:

$$t = t_m, \quad R = R_m, \quad \dot{R} \neq 0, \quad \ddot{R} = 0, \quad \text{then } \chi = \chi_m \quad (29)$$

Thus, eq. (28) takes the form:

$$\begin{aligned} & \int_0^\nu \frac{\chi'(\xi)}{(\nu-\xi)^{1/2}} d\xi \rightarrow \frac{1}{\lambda} \left[ \frac{(N-2)}{\varepsilon} \left( 1 - \frac{1}{\left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\}} \chi_f^{1/3} \right) - \right. \\ & \left. - \frac{(N-1)(N-2)R_0}{\left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \sigma} C \right] \end{aligned} \quad (30)$$

To integrate this term:

$$\int_0^\nu \frac{\chi'(\xi)}{(\nu-\xi)^{1/2}} d\xi$$

we suppose that  $\xi = z\nu$ , then we get:

$$\int_0^\nu \frac{\chi'(\xi)}{(\nu-\xi)^{1/2}} d\xi = \nu^{-1/2} \int_0^1 (1-z)^{1/2} \frac{d\chi(z\nu)}{dz} \frac{dz}{d\xi} \nu dz = \nu^{-1/2} \int_0^1 (1-z)^{-1/2} \frac{d\chi(z\nu)}{dz} dz \quad (31)$$

Since eq. (31) is independent of the time, we assume  $\chi = h_0 \nu^{1/2}$ , where  $h_0$  is a constant. Hence:

$$\chi(z\nu) = h_0(z\nu)^{1/2} \text{ and } \frac{d\chi(z\nu)}{dz} = \frac{1}{2}h_0\nu^{1/2}z^{-1/2} \quad (32)$$

The integration (31) can be reduced to:

$$\int_0^\nu \frac{\chi'(\xi)}{(\nu - \xi)^{1/2}} d\xi = \frac{h_0}{2} \int_0^1 (1 - z)^{-1/2} z^{-1/2} dz = \pi \frac{h_0}{2} \quad (33)$$

Using (33) into (30), we get:

$$h_0 \rightarrow \frac{2}{\pi\lambda} \left[ \frac{(N-2)}{\varepsilon} \left( 1 - \frac{1}{\left\{ 1 + \frac{\varepsilon\rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \chi_f^{1/3}} \right) - \frac{(N-1)(N-2)R_0}{\left\{ 1 + \frac{\varepsilon\rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \sigma} C \right] \quad (34)$$

Substituting from eq. (30) into this assumption  $\chi = h_0\nu^{1/2}$ , then we get:

$$\chi = \nu^{1/2} \frac{2}{\pi\lambda} \left[ \frac{(N-2)}{\varepsilon} \left( 1 - \frac{1}{\left\{ 1 + \frac{\varepsilon\rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \chi_f^{1/3}} \right) - \frac{(N-1)(N-2)R_0}{\left\{ 1 + \frac{\varepsilon\rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \sigma} C \right] \quad (35)$$

Combining eqs. (22) and (35) and using the value  $\lambda$  from eq. (28), then we obtain:

$$R = \left( \frac{12}{\pi} a_l \right)^{1/2} \text{Ja} \left( \left( 1 - \frac{1}{\left\{ 1 + \frac{\varepsilon\rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \chi_f^{1/3}} \right) - \frac{(N-1)R_0}{\left\{ 1 + \frac{\varepsilon\rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \sigma} C \right)^{1/2} t^{1/2} \quad (36)$$

Here the Jacob number, thermal diffusivity and initial void fraction are defined in the form  $\text{Ja} = (\rho_l C_P \Delta\theta_0)/(\rho_v L)$ ,  $a_l = k_l/(\rho_l C_P)$ , and  $\varphi_0 = (R_0/R_m)^3$ , respectively, and the condition  $0 < \varphi_0 < 1$  is satisfied.

By using the previous relations, eq. (36) reads:



$$R = \left( \frac{12}{\pi} a_l \right)^{1/2} \text{Ja} \left( 1 - \frac{\phi_0^{1/3}}{\left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\}} - \frac{(N-1)R_0}{\left\{ 1 + \frac{\varepsilon \rho R_0}{2\sigma} \left[ \frac{N}{2(N-2)} \dot{R}_0^2 + \frac{2(N-1)}{\rho} C \right] \right\} \sigma} C \right) t^{1/2} \quad (37)$$

Equation (37) represents the behavior of growth vapor bubble in  $N$ -dimensions fluid for the variable viscosity between two finite boundaries. The growth of bubble depends on initial void fraction,  $N$ -dimensions fluid and parameter  $C$ .

## Results and discussions

The Rayleigh-Plesset eq. (14) is derived in  $N$ -dimensions, that is included the effect of the viscosity, density ratio, pressure and surface tension. This equation is solved analytically using the modified method in [7]. eq. (14) is reduced to the model of Rayleigh-Plesset [7] when  $N = 3$ , and we can obtain the Rayleigh-Plesset [7] for considering viscosity in the form:

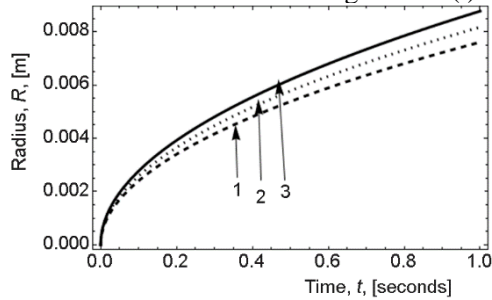
$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho}\Delta P - \frac{4\eta}{\rho} \frac{\dot{R}}{R}$$

Note that eq. (14) is satisfied for every value bigger than and equal three,  $N \geq 3$ . The velocity of the mixture of vapor and superheated liquid in  $N$ -dimensions is obtained in eq. (6). The general analytical solution of  $N$ -dimensions Rayleigh-Plesset equation for bubble growth is given in eq. (37). The obtained solution in eq. (37) represents the behavior the growth of bubble in  $N$ -dimensions fluid. The numerical values of physical parameters, which are used in the simulating and calculations for the bubble growth as:  $\rho = 937.5 \text{ kg/m}^3$ ,  $\rho_v = 0.579 \text{ kg/m}^3$ ,  $\sigma = 0.0535 \text{ m}^3/\text{sec}$ ,  $L = 533 \cdot 10^3 \text{ J/kg}$ ,  $\Delta\theta_0 = 1.0 \text{ }^\circ\text{K}$ ,  $\eta_0 = 0.89 \cdot 10^{-3} \text{ Pa}\cdot\text{s}$ ,  $C_{p,l} = 4240 \text{ J/kgK}$ ,  $C_{p,v} = 2160 \text{ J/kgK}$ , and  $k_l = 0.6857 \text{ W/mK}$ . These values of physical parameters are in [19], and  $R_0 = 10^{-5} \text{ m}$ ,  $R_m = 10^{-4} \text{ m}$ , and  $\dot{R}_0 = 0.2 \text{ m/s}$ . The void fraction  $\phi_0$  is estimated as  $\phi_0 = 1.0 \cdot 10^{-3}$ , and Jacob number is also estimated as  $\text{Ja} = 12.88$ .

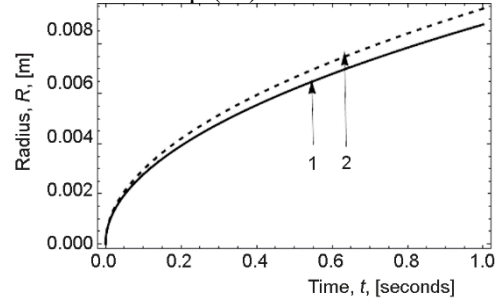
Figure 1 shows the results of the analytical solution of Rayleigh-Plesset equation in  $N$ -dimensions liquid for the growth of vapor bubble. We note that fig. 1 illustrates the radius of bubble for some different values of  $N$ -dimensions. In other words, we can say that we obtain the behavior of vapor bubble at  $N \geq 3$ . The  $N$  is denoted the  $N$ -dimensions spherical bubbles, this results are satisfied when we can find for certain value of  $N$  that means  $N = 3$ , in this case for  $N = 3$ , we can get the Rayleigh equation [6] and its solution for neglecting viscosity of liquid and surface tension. For the following the effect of  $N$  on bubble dynamic in fig. 1, we can illustrate that it can see, the radius and the period of the bubble decrease with the increasing of the space  $N$ -dimension. These results are agreement with the results in [14], although results in [14] have been obtained in empty-filled spherical cavity in an infinite volume.

Figure 2 refers the results of the behavior of vapor bubble in viscous and non-viscous liquid at  $N$ -dimensions (*i. e.*, we choose  $N = 3$ ), and we get the behavior of vapor bubble in viscous fluid is smaller than one. This result should be taken when we study cavitations and bubble dynamics that we remark, the liquid viscosity plays an important role on

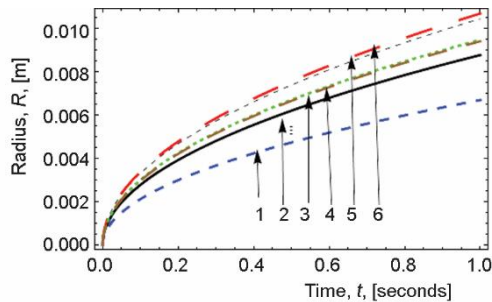
growth of bubbles. Figures 3(a) and 3(b) reflect the comparison between the present model for different values of  $N$  and several previous models in [7-9, 20, 21] for the growth of vapor bubbles in water at relatively high superheats. To the best of our knowledge, the presented solution of the radius of bubble growth  $R(t)$  in  $N$ -dimensions in eq. (37) has been obtained for



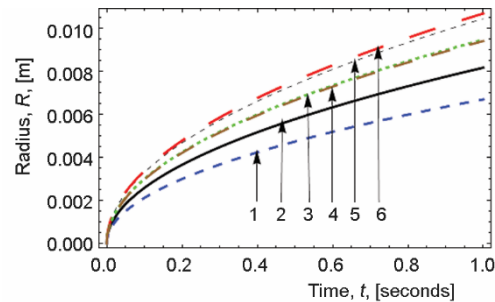
**Figure 1.** The behavior of bubble growth  $R(t)$  vs. the time,  $t$ , for some different of  $N$ -dimensions; 1 –  $N = 7$ , 2 –  $N = 5$ , 3 –  $N = 3$



**Figure 2.** The behavior of bubble growth  $R(t)$  vs. the time,  $t$ ; 1 – viscous fluid, 2 – non-viscous fluid;  $N = 3$



**Figure 3(a).** Comparison the present model with the previous models at  $N = 3$ ; 1 – [9], 2 – Results for present model, 3 – [21], 4 – [8], 5 – [7], 6 – [20]



**Figure 3(b).** Same the fig. 3(a) at  $N = 5$

the first time and we can reach to a special cases of the previous solutions of  $R(t)$ , i. e. in the models [7-9, 20, 21]. It is clear that, if we put  $R(t) = \Omega_1 \sqrt{t}$ , we can get:

- the solution of Plesset and Zwick model [7], at  $N = 3$ ,  $\varphi_0 = 0$ ,  $C = 0$ ,  $\Omega_1 = \text{Ja} \left( \frac{12}{\pi} a_l \right)^{1/2}$ ,
- the solution of Mohammadein model [8], at  $N = 3$ ,  $C = 0$ ,

$$\Omega_1 = \text{Ja} \left( \frac{12}{\pi} a_l \right)^{1/2} \left( 1 - \frac{4\pi\varphi_0^{1/3}}{3\dot{\rho}R_0^2R_0} \right)$$

- the solution of Olek model [9], at  $N = 3$ ,  $C = 0$ ,  $\varphi_0 = 0$ ,  $\Omega_1 = \frac{1 + \left( 1 - \frac{2\pi}{\text{Ja}} \right)}{\pi^{1/2}} \text{Ja} a_l^{1/2}$
- the solution of Forster model [20], at  $N = 3$ ,  $\varphi_0 = 0$ ,  $C = 0$ ,  $\Omega_1 = \pi \text{Ja} a_l^{(1/2)}$ , and
- the solution of Bosnjakovic model [21], at  $N = 3$ ,  $\varphi_0 = 0$ ,  $C = 0$ ,  $\Omega_1 = 2(\text{Ja} a_l)^{1/2}$ .

## Conclusion

We have studied the continuity equation and Rayleigh-Plesset equation in  $N$ -dimension for the description of a vapor/gas spherical bubble dynamics. We have analyzed the general solutions of the Rayleigh-Plesset equation in  $N$ -space dimensions for the terms of surface tension and viscosity. We have discussed dependence of these solutions on  $N$ -dimensions. We have also considered influence of the viscosity on the bubble motion. We concluded that the bubble radius decrease with the increasing of the space  $N$ -dimensions. The growth of vapor bubble in viscous liquid is smaller than in non-viscous liquid. These conclusions should be used in the applications of bubble dynamics

## Nomenclature

$A$	– constants in eq. (16), [–]
$a_l$	– thermal diffusivity, [m <sup>2</sup> s <sup>–1</sup> ]
$c_1$	– constants in eq. (5), [–]
$C_p$	– specific heat capacity, [Jkg <sup>–1</sup> K <sup>–1</sup> ]
$d_1, d_2, d_3$	– constants in eq. 20, [–]
$h_0$	– constant in eq. (32), [–]
$Ja$	– Jacob number
$k_l$	– thermal conductivity, [kgm <sup>–1</sup> s <sup>–1</sup> ]
$L$	– latent heat, [Jkg <sup>–1</sup> ]
$N$	– $N$ -dimensions fluid in eq. (2), [–]
$r$	– radial co-ordinate, [m]
$\Delta P$	– difference between the external pressure at infinity and in mixture, [kgm <sup>–1</sup> s <sup>–2</sup> ]
$R$	– radius of bubble, [m]
$\dot{R}$	– velocity of bubble, [ms <sup>–1</sup> ]
$\ddot{R}$	– acceleration of bubble, [ms <sup>–2</sup> ]
$T$	– temperature, [K]
$t$	– time, [s]
$u$	– velocity in mixture, [ms <sup>–1</sup> ]
$y_1, y_2$	– constants in eq. (21)
$z$	– variables in eq. (31)

## Greek symbols

$\gamma$	– constants in eq. (22), [–]
$\chi$	– dimensionless parameter defined by eq. (22), [–]
$\varepsilon$	– ratio of density [–]
$\sigma$	– surface tension, [m <sup>3</sup> s <sup>–1</sup> ]
$\phi$	– void fraction
$\Delta\theta$	– superheating liquid, [K]
$\eta$	– viscosity, [Pa·s]
$\lambda$	– constant is defined by eq. (28)
$\rho, \rho_v$	– density [kgm <sup>–3</sup> ]
$\nu$	– constants in eq. (22), [–]
$\zeta$	– variables in eq. (31)

## Subscripts

0	– initial value
m	– maximum value
v	– vapor
l	– liquid
$\infty$	– infinity value

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